NAG Toolbox for MATLAB

s21cc

1 Purpose

s21cc returns the value of one of the Jacobian theta functions $\theta_0(x,q)$, $\theta_1(x,q)$, $\theta_2(x,q)$, $\theta_3(x,q)$ or $\theta_4(x,q)$ for a real argument x and nonnegative $q < 1$, via the function name.

2 Syntax

```
[result, ifail] = s21cc(k, x, q)
```

3 Description

s21cc evaluates an approximation to the Jacobian theta functions $\theta_0(x,q)$, $\theta_1(x,q)$, $\theta_2(x,q)$, $\theta_3(x,q)$ and $\theta_4(x,q)$ given by

$$\begin{aligned}
\theta_0(x,q) &= 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2n\pi x),\\
\theta_1(x,q) &= 2\sum_{n=0}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}\sin\{(2n+1)\pi x\},\\
\theta_2(x,q) &= 2\sum_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2}\cos\{(2n+1)\pi x\},\\
\theta_3(x,q) &= 1+2\sum_{n=1}^{\infty} q^{n^2}\cos(2n\pi x),\\
\theta_4(x,q) &= \theta_0(x,q),
\end{aligned}$$

where x and q (the *nome*) are real with $0 \le q < 1$.

These functions are important in practice because every one of the Jacobian elliptic functions (see s21cb) can be expressed as the ratio of two Jacobian theta functions (see Whittaker and Watson 1990). There is also a bewildering variety of notations used in the literature to define them. Some authors (e.g., Section 16.27 of Abramowitz and Stegun 1972) define the argument in the trigonometric terms to be x instead of πx. This can often lead to confusion, so great care must therefore be exercised when consulting the literature. Further details (including various relations and identities) can be found in the references.

s21cc is based on a truncated series approach. If t differs from x or $-x$ by an integer when $0 \le t \le \frac{1}{2}$, it follows from the periodicity and symmetry properties of the functions that $\theta_1(x,q) = \pm\theta_1(t,q)$ and $\theta_3(x,q) = \pm\theta_3(t,q)$. In a region for which the approximation is sufficiently accurate, θ_1 is set equal to the first term ($n = 0$) of the transformed series

$$\theta_1(t,q) = 2\sqrt{\frac{\lambda}{\pi}}e^{-\lambda t^2}\sum_{n=0}^{\infty}(-1)^n e^{-\lambda\left(n+\frac{1}{2}\right)^2}\sinh\{(2n+1)\lambda t\}$$

and θ_3 is set equal to the first two terms (i.e., $n \le 1$) of

$$\theta_3(t,q) = \sqrt{\frac{\lambda}{\pi}}e^{-\lambda t^2}\left\{1+2\sum_{n=1}^{\infty}e^{-\lambda n^2}\cosh(2n\lambda t)\right\},$$

where $\lambda = \pi^2/|\log_e q|$. Otherwise, the trigonometric series for $\theta_1(t,q)$ and $\theta_3(t,q)$ are used. For all values of x, θ_0 and θ_2 are computed from the relations $\theta_0(x,q) = \theta_3\left(\frac{1}{2}-|x|,q\right)$ and $\theta_2(x,q) = \theta_1\left(\frac{1}{2}-|x|,q\right)$.

4 References

Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (3rd Edition) Dover Publications

Byrd P F and Friedman M D 1971 *Handbook of Elliptic Integrals for Engineers and Scientists* pp. 315–320 (2nd Edition) Springer–Verlag

Magnus W, Oberhettinger F and Soni R P 1966 *Formulas and Theorems for the Special Functions of Mathematical Physics* 371–377 Springer–Verlag

Tølke F 1966 *Praktische Funktionenlehre (Bd. II)* 1–38 Springer–Verlag

Whittaker E T and Watson G N 1990 *A Course in Modern Analysis* (4th Edition) Cambridge University Press

5 Parameters

5.1 Compulsory Input Parameters

1: **k – int32 scalar**

The function $\theta_{\mathbf{k}(x,q)}$ to be evaluated. Note that $\mathbf{k} = 4$ is equivalent to $\mathbf{k} = 0$.

Constraint: $0 \leq \mathbf{k} \leq 4$.

2: **x – double scalar**

The argument x of the function.

3: **q – double scalar**

The argument q of the function.

Constraint: $0.0 \leq \mathbf{q} < 1.0$.

5.2 Optional Input Parameters

None.

5.3 Input Parameters Omitted from the MATLAB Interface

None.

5.4 Output Parameters

1: **result – double scalar**

The result of the function.

2: **ifail – int32 scalar**

0 unless the function detects an error (see Section 6).

6 Error Indicators and Warnings

Errors or warnings detected by the function:

$\mathbf{ifail} = 1$

On entry, $\mathbf{k} < 0$,
or $\mathbf{k} > 4$,
or $\mathbf{q} < 0.0$,
or $\mathbf{q} \geq 1.0$,

ifail = 2

The evaluation has been abandoned because the function value is infinite. The result is returned as the largest machine representable number (see x02al).

7 Accuracy

In principle the function is capable of achieving full relative precision in the computed values. However, the accuracy obtainable in practice depends on the accuracy of the standard elementary functions such as SIN and COS.

8 Further Comments

None.

9 Example

```
k = int32(2);
x = 0.7;
q = 0.4;
[result, ifail] = s21cc(k, x, q)
```

```
result =
   -0.6929
ifail =
           0
```